

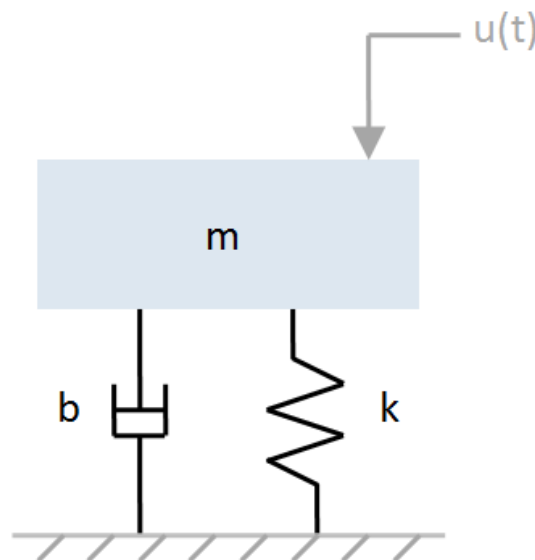
System Identification for a Mass Spring Damper

▼ Introduction

A mass spring damper is disturbed by an input force, with its response recorded. The data is recorded in a spreadsheet.

This application will

- import the time-based experimental data from the spreadsheet
- translate the experimental data to the frequency domain
- and then identify the mass, damping coefficient and spring constant via system identification of a model transfer function



```
> restart :  
with( DynamicSystems ) :  
with( SignalProcessing ) :  
with( ExcelTools ) :  
with( plots ) :  
with( Optimization ) :  
with( CurveFitting ) :
```

▼ Model Function

$$> tf := \frac{1}{M \cdot s^2 + b \cdot s + k} :$$

$$> MassSpringDamperFrequency := subs(s = 2 \cdot i \cdot \pi \cdot \omega, tf)$$

$$MassSpringDamperFrequency := \frac{1}{-4 M \pi^2 \omega^2 + 2 i b \pi \omega + k} \quad (2.1)$$

$$> RealPart := unapply(evalc(\Re(MassSpringDamperFrequency)), \omega)$$

$$RealPart := \omega \mapsto \frac{-4 M \pi^2 \omega^2 + k}{(-4 M \pi^2 \omega^2 + k)^2 + 4 b^2 \pi^2 \omega^2} \quad (2.2)$$

$$> ImaginaryPart := unapply(evalc(\Im(MassSpringDamperFrequency)), \omega)$$

$$ImaginaryPart := \omega \mapsto -\frac{2 b \pi \omega}{(-4 M \pi^2 \omega^2 + k)^2 + 4 b^2 \pi^2 \omega^2} \quad (2.3)$$

▼ Import Experimental Data and Convert to Frequency Domain

$$> data := ImportMatrix("this:///ResponseData.xlsx")$$

$$data :=$$

$$(3.1)$$

0.	0.0549733508585818	0.
0.0500000000000000	0.0543561947202864	0.
0.1000000000000000	0.0585754216137588	$-6.43421170920021 \cdot 10^{-6}$
0.1500000000000000	-0.00222207860248313	-0.0000112345738620539
0.2000000000000000	-0.0224170308129334	0.0000431292951339169
0.2500000000000000	0.0470538171872169	0.000107268246427184
0.3000000000000000	0.0205527623858366	0.000210952378670105
0.3500000000000000	-0.00529459731192777	0.000329531753997433
0.4000000000000000	0.0519881729708760	0.000464083572993199
0.4500000000000000	0.101888561624161	0.000642448983069600
\vdots	\vdots	\vdots

1000 × 3 Matrix

$$> SignalTime := data[., 1] :$$

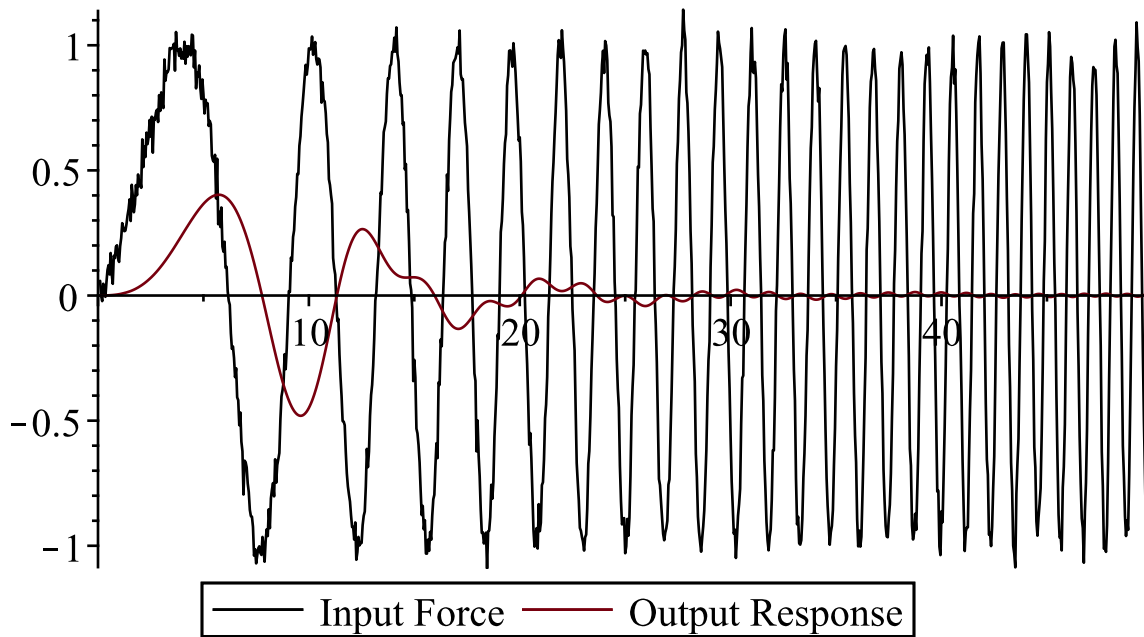
$$InputForce := data[., 2] :$$

$$OutputResponse := data[., 3] :$$

$$> N := 1000 :$$

$$samplingRate := \frac{1}{SignalTime[2] - SignalTime[1]} :$$

```
> p1 := plot( SignalTime, InputForce, legend = "Input Force", color = black, thickness = 0 ) :
p2 := plot( SignalTime, OutputResponse, legend = "Output Response", thickness = 0 ) :
display( p1, p2, size = [ 600, 250 ] )
```



Discrete Fourier Transform of input and output

```
> InputDFT := DFT( InputForce )
```

```
InputDFT := [ 1.42186485453318 + 0.1, 1.30870684189390 - 0.417511675327161 I,
```

(3.2)

```
1.25739329172986 - 0.997267005429405 I, 0.854548633690916
- 1.45378547970901 I, 0.267465685322502 - 1.90483547315609 I,
- 0.745797530416493 - 2.02257166492739 I, - 1.76000056628522
- 1.44499250465225 I, - 2.38686225248846 - 0.0848382857274335 I,
- 1.83682286345747 + 1.57556784145005 I, 0.134609314859537
+ 2.26826592143957 I, 1.94690392865862 + 0.849127786296443 I, 1.39896257717207
- 1.56161107891914 I, - 1.36067469740546 - 1.84106487391668 I,
- 2.22792767266118 + 0.958531478950645 I, 0.795845835191632
+ 2.06675260451282 I, 1.74790289799883 - 1.16104513542706 I,
- 1.71517610258981 - 1.56520833127439 I, - 0.940335076996829
+ 2.07017495042380 I, 2.10753859175297 - 0.289551711348798 I,
- 1.63266069643198 - 1.62594793287254 I, - 0.295029463909527
+ 2.28360042418173 I, 1.33660717748546 - 1.56235408613733 I,
- 2.32970661394607 + 0.511264934333504 I, 2.01206752576985
+ 0.492338784773037 I, - 2.08821021988255 - 1.09231972760551 I,
1.58789829968027 + 1.47174733377688 I, - 1.75753820353273
- 1.57501509830175 I, 1.61368944271928 + 1.42474797841998 I,
```

−2.14106025778732 − 1.01926862509486 I, 2.07405347703486
+ 0.474567325654122 I, −2.32968583537705 + 0.566821094817876 I,
1.35951038449027 − 1.63874080332448 I, −0.339550617965056
+ 2.22068894180865 I, −1.79241105642943 − 1.65524280958019 I,
2.03495734988511 − 0.318532187973825 I, −1.02451965403208
+ 2.10794693396313 I, −1.81929019228236 − 1.46402909290448 I,
1.64847865216390 − 1.21742913655604 I, 0.716431535013009 + 2.04439620150022 I,
−2.25552141444711 + 0.958824414525206 I, −1.50486204338471
− 1.79987615205460 I, 1.24328844794459 − 1.57128026841088 I, 1.80809752091938
+ 0.787120730621837 I, 0.0668916964077781 + 2.21567349249208 I,
−1.97512268264223 + 1.64437784143976 I, −2.62029039941388
− 0.0192841760815246 I, −2.14354576044591 − 1.38900511109316 I,
−1.14541828639919 − 2.05133617759493 I, −0.272009358165653
− 1.96039614647991 I, 0.374977080663978 − 1.62771278868780 I,
0.605854232796672 − 1.22244122543958 I, 0.713162340026646
− 0.888845264572108 I, 0.631877087421227 − 0.572803627081779 I,
0.558830705767032 − 0.362846386685171 I, 0.513090606553623
− 0.221486996382308 I, 0.487943275413511 − 0.155539870651046 I,
0.448558213837338 − 0.179532894448736 I, 0.430289915932281
− 0.0855807452410588 I, 0.287351602796811 − 0.126662079920024 I,
0.321967688208054 − 0.0295287039414501 I, 0.294031622200415
− 0.0905151232586377 I, 0.241099225711075 + 0.00661461683719412 I,
0.205640604458355 + 0.0102496606505561 I, 0.195016002785927
− 0.0346153518947048 I, 0.141754327768445 + 0.0568377221622173 I,
0.134648268101419 − 0.00589435844171353 I, 0.143989692832845
− 0.0285505398130136 I, 0.165297699570615 − 0.0452082745062494 I,
0.133482798280664 + 0.0211604248971377 I, 0.138295529480239
− 0.0528313124792853 I, 0.0987996134896791 − 0.000844657364879202 I,
0.112136768203029 − 0.0553138812120639 I, 0.0999588286625177
− 0.0440164580243498 I, 0.0891160797452222 − 0.0295559661493706 I,
0.0869758725362109 + 0.0258927556264003 I, 0.0765225482389859
− 0.0705414987403152 I, 0.0856923244756928 − 0.00890183287890250 I,
0.0892401424487809 − 0.0278301324528985 I, 0.141447791801527
− 0.0500138970251390 I, 0.0920914129987321 − 0.00526381543236250 I,
0.0538096502209863 − 0.0397700371651849 I, 0.0743806116965889
− 0.00316154600354388 I, 0.115509534440098 + 0.00746298936854114 I,
0.0672330771932313 + 0.0142644022671210 I, 0.123790463667166
+ 0.0144214130919593 I, 0.0863412970446363 − 0.00210180171679712 I,

0.0978287534205901 + 0.0516634278035433 I, -0.00847944997174084
+ 0.0113542626947054 I, 0.0259385702549860 + 0.00737106271202121 I,
0.0700155399615564 + 0.0100218542576911 I, 0.0444224398227368
+ 0.0234107971961277 I, 0.0664362381148397 - 0.000195246214795274 I,
0.102519587290319 - 0.0593647537234550 I, 0.0604580552362261
- 0.0184283323000477 I, 0.00318131870241016 + 0.00944109053881416 I,
-0.0203924618692019 + 0.0401884467581884 I, -0.00429303036937164
- 0.0140854920555922 I, 0.0368771271209320 + 0.0314102731772076 I,
0.0304736821507428 - 0.0668062670070301 I, 0.0738583615820692
- 0.0667012699794676 I,..., ... 900 Array entries not shown]

> *OutputDFT* := *DFT*(*OutputResponse*)

OutputDFT := [0.504463515362046 + 0. I, 0.490485238479234 - 0.194040669594530 I,

(3.3)

0.397313208428457 - 0.397049888199579 I, 0.215841722422724
- 0.699012229388557 I, -0.289089417495720 - 0.911992798689711 I,
-1.21349726883948 - 0.538422196837943 I, -1.13144703496336
+ 1.12675768047248 I, 0.570386465316985 + 1.11329212852861 I,
0.847590665084104 - 0.0220141404152165 I, 0.245946246346201
- 0.534949028321740 I, -0.296145587843951 - 0.299062820789514 I,
-0.267212010705840 + 0.158369226673304 I, 0.0779523107794714
+ 0.239826124337035 I, 0.208196696349861 - 0.0452815557154894 I,
-0.0452400327858703 - 0.171575128779050 I, -0.135381431009969
+ 0.0543632417685608 I, 0.0809206912738270 + 0.0985460426224233 I,
0.0538231435649861 - 0.102310170999757 I, -0.0972410661842284
+ 0.00601500098487299 I, 0.0564137257661061 + 0.0698509489669909 I,
0.0108348069770875 - 0.0789398617869015 I, -0.0490348076503301
+ 0.0493632426858788 I, 0.0611103929995075 - 0.0137058618865371 I,
-0.0542729749348762 - 0.00978330866846719 I, 0.0456904277378366
+ 0.0253049728935012 I, -0.0371731474788827 - 0.0304584690481361 I,
0.0341191819805991 + 0.0292111825937995 I, -0.0320530025488332
- 0.0241843225730665 I, 0.0361696354951711 + 0.0157927939355376 I,
-0.0346357747683778 - 0.00441958911425017 I, 0.0321116304643193
- 0.0106839133599923 I, -0.0174559396696812 + 0.0250138321605742 I,
-0.000913509526792102 - 0.0293165787023756 I, 0.0215680742215969
+ 0.0176668382562808 I, -0.0239545804972594 + 0.00651257478287539 I,
0.00628091359703390 - 0.0230255389017857 I, 0.0198352067569141
+ 0.0128415442707395 I, -0.0152291868143240 + 0.0146965575114250 I,
-0.0109597418196850 - 0.0173137577652648 I, 0.0168165569813947
- 0.0113798253553161 I, 0.0138992363592371 + 0.0132840823126126 I,

—0.00782965378847197 + 0.0143049102327398I, —0.0153099714423534
— 0.00337505951084481I, —0.00382810986324710 — 0.0153462021705993I,
0.00982548432870395 — 0.0137323641636426I, 0.0161105828295524
— 0.00334118230413097I, 0.0140383291881571 + 0.00551830497789170I,
0.00858825105821143 + 0.0102885334873883I, 0.00332221420238915
+ 0.0106816776373290I, 0.0000258069924856856 + 0.00892554656369747I,
—0.00205945893177935 + 0.00691034942502993I, —0.00267939214317340
+ 0.00513891597011221I, —0.00306228838808445 + 0.00364908792264396I,
—0.00275718477391920 + 0.00272730576755861I, —0.00273041907882688
+ 0.00236643015886292I, —0.00224646845972580 + 0.00166183885839677I,
—0.00205952701497002 + 0.00113850687839880I, —0.00197058212416834
+ 0.00112306390939316I, —0.00157864596105923 + 0.000821708571551499I,
—0.00154875603745316 + 0.000842361992664961I, —0.00121193994912708
+ 0.000648926278488011I, —0.00127994483289957 + 0.000810707149830058I,
—0.000888063380898822 + 0.000276480206655517I, —0.000922629935678135
+ 0.000618117147640316I, —0.000989792750153316 + 0.000430230759082821I,
—0.000985486471157991 + 0.000397202861419471I, —0.000723554404916331
+ 0.000542259920146270I, —0.000732644703313493 + 0.000348013684180913I,
—0.000594900911233418 + 0.000345317138524550I, —0.000613944281849488
+ 0.000212456130886389I, —0.000693045152612816 + 0.000276191863213991I,
—0.000660797904607681 + 0.000146494439810397I, —0.000398183699312589
+ 0.000548666471279883I, —0.000414706784443027 + 0.000190413628241920I,
—0.000343398889038199 + 0.000447985794234302I, —0.000337348696190460
+ 0.000262157491691858I, —0.000290221595563186 + 0.000261962057039003I,
—0.000421687582306532 + 0.000355827695398597I, —0.000346196633081392
+ 0.000197137968033831I, —0.000466450463129339 + 0.000247665950613010I,
—0.000253196884579535 + 0.000175857083467241I, —0.000336204323250379
+ 0.000149322393009304I, —0.000365549786553767 + 0.000200661637924035I,
—0.000346522916077015 + 0.000123516338498261I, —0.000277688058093469
+ 0.000107524449016480I, —0.000234223300077777 + 0.000159421714224175I,
—0.000314212294452078 + 0.000177738319255950I, —0.000285048411573804
+ 0.000144499992779196I, —0.000326250998591368 + 0.000153765873080771I,
—0.000235629515723360 + 0.0000706001652963714I, —0.000290670302171509
+ 0.000191000236259853I, —0.000287047380877274 + 0.000231944980522217I,
—0.000208473687979060 + 0.000156815906519107I, —0.000331681300407009
+ 0.000191527449676221I, —0.000171765290211543 + 0.000145258459804981I,
—0.000330875560761936 + 0.000140932041793094I, —0.000297398855597715
+ 0.000185717494463971I, —0.000314429384507192 + 0.000125585430794513I,

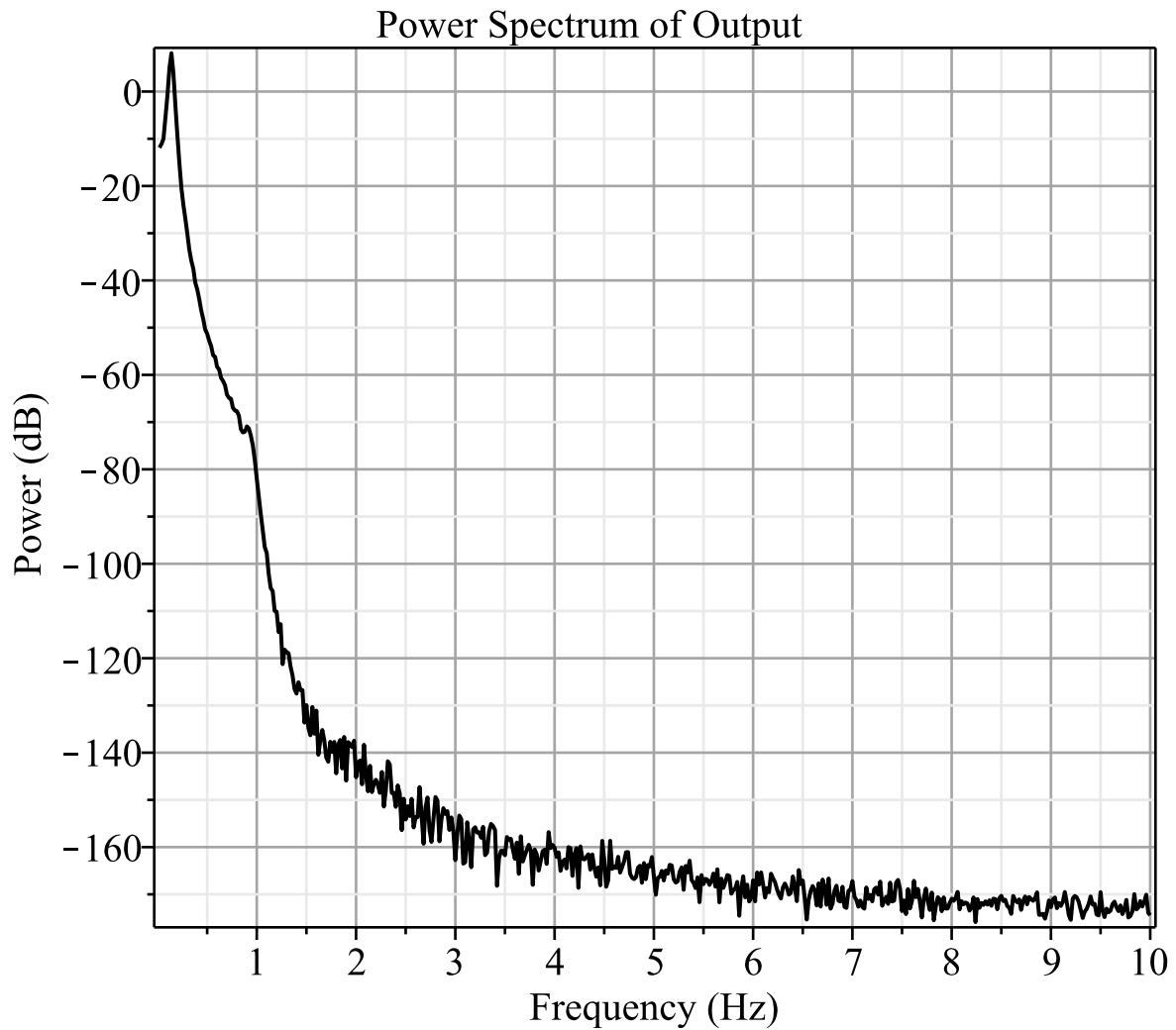
$-0.000297697986389620 + 0.000212635306343198i$, -0.000137656556928527
 $+ 0.000189549773428207i$, ..., ... 900 Array entries not shown]

Power spectrum of the output response

> *OutputPS* := *PowerSpectrum*(*OutputDFT*)

$$\begin{array}{lcl}
 \text{OutputPS} := & \left[\begin{array}{c} 0.254483438331434 \\ 0.278227550622724 \\ 0.315506399311013 \\ 0.535205745973169 \\ 0.915303556169906 \\ 1.76247408352867 \\ 2.54975526343110 \\ 1.56476008326058 \\ 0.718894557915934 \\ 0.346660018994159 \\ \vdots \end{array} \right] & (3.4) \\
 & \text{1000 element Vector[column]} &
 \end{array}$$

> *pointplot*([*seq*([$i \cdot \text{samplingRate} / N$, $20 \cdot \log_{10}(\text{OutputPS}_i)$], $i = 1 \dots N/2$)], *connect* = *true*, *title*
 = "Power Spectrum of Output", *labels* = ["Frequency (Hz)", "Power (dB)"], *gridlines*,
labeldirections = [*horizontal*, *vertical*], *axes* = *boxed*, *size* = [800, 400])



> *MassSpringDamperDFT* := *OutputDFT*/~ *InputDFT*

MassSpringDamperDFT := [0.354790058811650 + 0.1, 0.383097135104929

(3.5)

– 0.02605101602013221, 0.347705734237602 – 0.03999896625551511,

0.422211144038435 – 0.09971088297938871, 0.448622544405808

– 0.2147589960137321, 0.429097571149086 – 0.4417531281893291,

0.0700383072321674 – 0.6977057467960441, –0.255225257362513

– 0.4573532695829141, –0.271767142803258 – 0.2211282526088631,

–0.228600758168013 – 0.1219954129557911, –0.184089156402191

– 0.07332031171712321, –0.141303352214035 – 0.04452703356687761,

–0.104485372948487 – 0.03488105897316361, –0.0862313267967137

– 0.01677517899232301, –0.0796371912440059 – 0.008776519333966001,

–0.0680760424496850 – 0.01411755549179751, –0.0543498315465512

– 0.007857638327663201, –0.0507580530545253 – 0.002943502836077941,

–0.0456697058915661 – 0.003420454809717281, –0.0387393739442851

– 0.004203411031148231, –0.0346035597987285 – 0.0002740134717445511,

−0.0337467024609800 − 0.00251461749892507l, −0.0262574423379423
+ 0.000120767289829142l, −0.0265724903851893 + 0.00163978540178850l,
−0.0221564850898670 − 0.000528206942452461l, −0.0221557775758908
+ 0.00135351081827311l, −0.0190271420272271 + 0.000430632674210880l,
−0.0185977575504755 + 0.00143323420163243l, −0.0166348350250692
+ 0.000542988682787455l, −0.0163320696600610 + 0.00160607117589082l,
−0.0140667568659013 + 0.00116349543426169l, −0.0142757899664924
+ 0.00119124698219943l, −0.0128385135141963 + 0.00237441391498788l,
−0.0114071410442328 + 0.000677718389816506l, −0.0119789851718365
+ 0.00132527712511348l, −0.0100074139774708 + 0.00188424036911250l,
−0.0100649915223352 + 0.00104100827115453l, −0.0102383196667567
+ 0.00135405383430997l, −0.00921577928721258 + 0.00213132215536733l,
−0.00813111810701410 + 0.00158877267755908l, −0.00814404090000859
+ 0.000913161901259310l, −0.00802342362016128 + 0.00136562276926601l,
−0.00780155584660237 + 0.00152962259785280l, −0.00697200826073335
+ 0.00151725442160545l, −0.00635687695467876 + 0.00166027993472431l,
−0.00613867912977893 + 0.00132029704575930l, −0.00578725312389109
+ 0.00117572446405745l, −0.00560554116869894 + 0.00105665844637980l,
−0.00557651273270795 + 0.000920911090548896l, −0.00520367528116223
+ 0.00121462826358515l, −0.00520850642455645 + 0.000896678473595059l,
−0.00498868865160838 + 0.000988195318444826l, −0.00553389060166756
+ 0.000758462877789913l, −0.00569973372061329 + 0.00117956650406618l,
−0.00616385560020822 + 0.00195134403758354l, −0.00516480368189674
+ 0.00175943804338519l, −0.00483307898594552 + 0.000603734834406222l,
−0.00490474991106003 + 0.00163450671452480l, −0.00565542909847235
+ 0.000366728976184908l, −0.00500810804735140 + 0.00215698369203793l,
−0.00438559866868882 + 0.000856925770579139l, −0.00521261288303445
+ 0.00350555496094706l, −0.00424097326820796 + 0.00155586365992968l,
−0.00513195757776985 + 0.00225864864362250l, −0.00496698748117307
+ 0.00502660500556205l, −0.00743385902802236 + 0.00262450484277276l,
−0.00555342833792625 + 0.00266481951406776l, −0.00465955100679246
+ 0.000831006260503265l, −0.00394745792511631 + 0.00321275123841712l,
−0.00438613671486323 − 0.000139333704754132l, −0.00703803915860087
+ 0.00273530555494128l, −0.00525786388946709 − 0.00128716406860821l,
−0.00536098355949263 + 0.00312823757189043l, −0.00483083229129019
+ 0.000534513105866449l, −0.00221825199081129 + 0.00581106502541531l,
−0.00409052513918720 − 0.000344921632495988l, −0.00366480422927350
+ 0.00267630250036184l, −0.00543974077779819 + 0.00229088595595169l,

-0.00261356493327381 + 0.000469596589289724i, -0.00520180552458989
 + 0.00239202113686276i, -0.00460524011565563 - 0.000135542361956088i,
 -0.00459707843179077 + 0.00181214586692462i, -0.00303974457427027
 + 0.00193358254318581i, -0.00455906226231500 + 0.00280440289869336i,
 -0.00211333765875864 + 0.00111480125615554i, -0.00275607491189606
 + 0.00177932225367553i, -0.00176120712190147 + 0.00274692569259329i,
 0.0202058434762134 + 0.0100150908727392i, -0.0100792783680559
 + 0.00879234529049614i, -0.00315638686184434 + 0.00146014748244175i,
 -0.00334764575015352 + 0.00606385630947049i, -0.00433086754705755
 + 0.00347851416010948i, -0.00218619158594456 + 0.000263687965799795i,
 -0.00590328844393012 + 0.00136854697434708i, 0.00831150383204580
 + 0.0209940612244652i, 0.00611099390151211 + 0.00513225484636079i,
 -0.00617614069112584 - 0.0229962300125419i, -0.00326038319525986
 + 0.00618255746637565i, -0.00431723338271297 - 0.00248683238799686i,
 -0.00230311277472929 + 0.000486461189744966i,..., ... 900 Array entries not shown]

▼ Optimization

Frequencies to sweep over

> *MinFreq* := 0.01 :
MaxFreq := 1 :

Hence the minimum and maximum indices to sweep over are

> *min_idx* := round($\frac{MinFreq \cdot N}{samplingRate}$) :
max_idx := round($\frac{MaxFreq \cdot N}{samplingRate}$) :

> *RealObjective* := add(*i*, i in (seq((*RealPart*(*i*/*max_idx*) - evalc($\Re(MassSpringDamperDFT[i + 1])$))², *i* = *min_idx*..*max_idx*)) :
ImaginaryObjective := add(*i*, i in (seq((*ImaginaryPart*(*i*/*max_idx*)
 - evalc($\Im(MassSpringDamperDFT[i + 1])$))², *i* = *min_idx*..*max_idx*)) :

> *opt* := Minimize(*RealObjective* + *ImaginaryObjective*)
opt := [0.00772982838513534571, [*M* = 5.13288427786581, *b* = 1.85057049009481, *k*
 = 3.07733924132606]] (4.1)

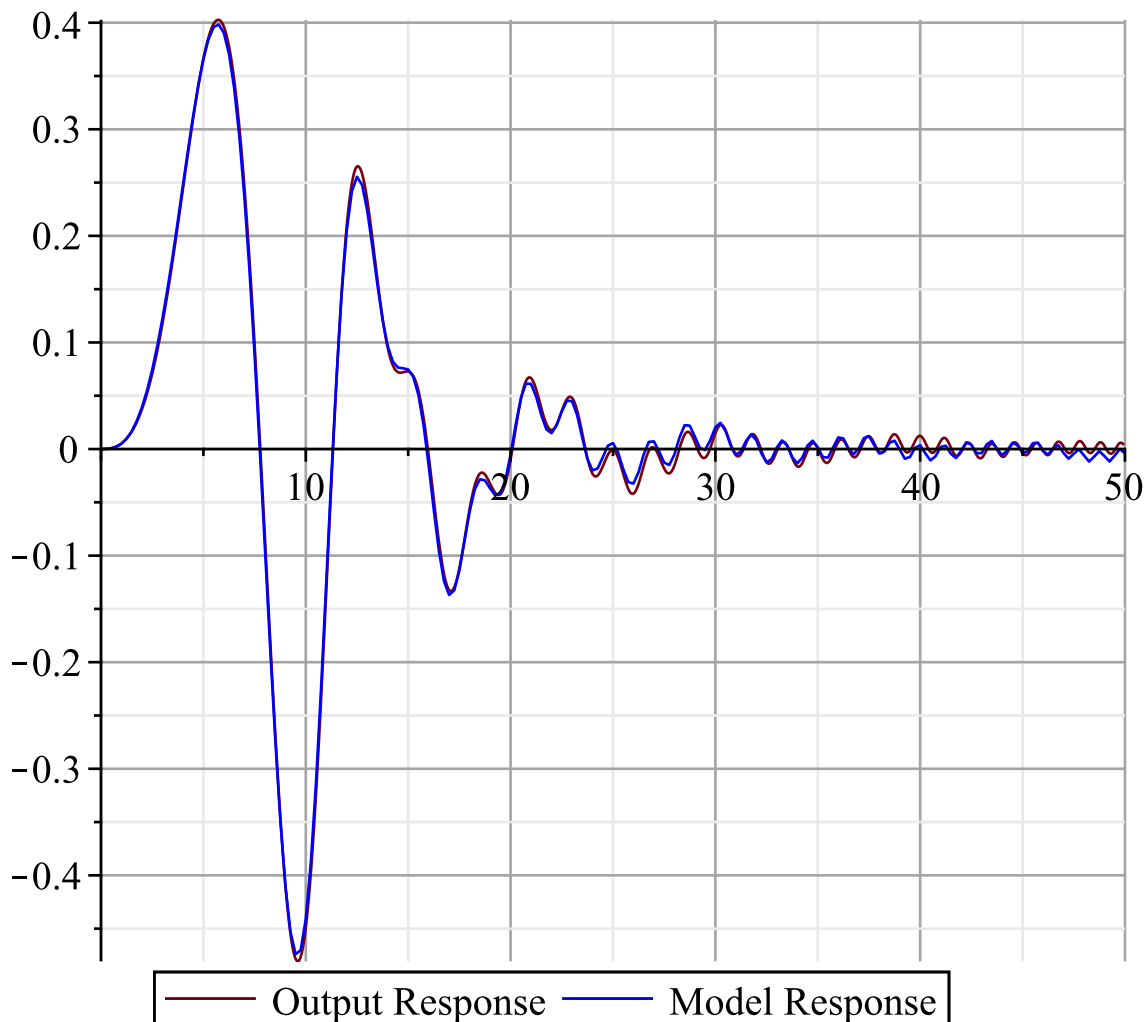
▼ Compare Time Domain Response of Experimental Output Data and Model Response

$$\begin{aligned}
 &> de := eval\left(M \frac{d^2}{dt^2} y(t) + b \frac{d}{dt} y(t) + ky(t) = u(t), opt[2]\right) \\
 de &:= 5.13288427786581 \frac{d^2}{dt^2} y(t) + 1.85057049009481 \frac{d}{dt} y(t) \\
 &\quad + 3.07733924132606 y(t) = u(t)
 \end{aligned}
 \tag{5.1}$$

```

> u := unapply(Spline(SignalTime, InputForce, t), t) :
> ic := y(0) = 0, y'(0) = 0 :
> de := dsolve({de, ic}, numeric)
      de := proc(x_rkf45) ... end proc
      (5.2)
> p3 := odeplot(de, 0..50, legend = "Model Response", color = blue, thickness = 0) :
      display(p2, p3, size = [800, 400], gridlines)

```



>

The model response closely matches the experimental results